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BEHAVIOUR OF A THICK CIRCULAR SLAB AFTER BUCKLING*

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Using the equations of the three-dimensional non-linear theory of elasticity, the problem of the axisymmetric buckling and initial post-critical behaviour of a circular cylinder of a neo-Hookean material compressed uniformly along the lateral surface is investigated. The cylinder endfaces are free while the lateral surface is clamped from rotation but can slide freely in the direction of the cylinder axis. Bifurcation of the cylinder equilibrium mode that occurs during attainment of critical values of the loading parameter is studied. Asymptotic representations are found for the branching solutions under almost critical loads. The qualitative distinction between the post-critical behaviour of a thick slab and the behaviour of a thin plate is disclosed.

1. Consider the equilibrium of an elastic circular cylinder $0 \leqslant r \leqslant a$, $-h \leqslant z \leqslant h$, loaded along the lateral surface. When there are no mass forces, the differential equations equilibrium have the form

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{\partial}{r\,\partial\theta} + \mathbf{e}_z \frac{\partial}{\partial z} \tag{1.1}$$

Here $\mathbf{D}$ is the non-symmetric Piola stress tensor, r, θ, z are cylindrical coordinates in the undeformed state of the body and $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are unit vectors tangent to the coordinate lines. It is assumed that a constant radial displacement is given on the cylinder lateral surface, and there are no shear stresses, while the plane faces of the body are stress-free. This results in the following boundary conditions

$$\mathbf{e}_z \cdot \mathbf{D} = 0, \quad z = \pm h \tag{1.2}$$
$$\mathbf{e}_r \cdot \mathbf{D} \cdot \mathbf{e}_z = 0, \quad \mathbf{R} \cdot \mathbf{e}_r = (1 - \varepsilon)\, a, \quad r = a$$

where $2h$ is the height of the cylinder, a is its radius in the undeformed state, $\mathbf{R}$ is the radius-vector of points of the deformed body, and ε is a loading parameter. For an incompressible neo-Hookean material we have /1, 2/

$$\mathbf{D} = 2c_1 \nabla \mathbf{R} + 2q\, (\nabla \mathbf{R}^T)^{-1} \tag{1.3}$$
$$\det (\nabla \mathbf{R}) = 1 \tag{1.4}$$

Here c_1 is a material constant, and q is an unknown function of the coordinates determined from the equilibrium equation and the incompressibility condition (1.4). The fundamental solution of the boundary value problem (1.1)-(1.4) that describes the precritical state of the cylinder in uniform strain and is given by the relationship

$$\mathbf{R}^\circ = \beta r \mathbf{e}_r + \beta^{-2} z \mathbf{e}_z, \quad q^\circ = -c_1 \beta^{-4}, \quad \beta = 1 - \varepsilon \tag{1.5}$$

Here and below the superscript $^\circ$ refers to the precritical state.

We shall seek axisymmetric equilibrium modes close to the fundamental solution, i.e., we set

$$\mathbf{R} = \mathbf{R}^\circ + u\,(r, z)\, \mathbf{e}_r + w\,(r, z)\, \mathbf{e}_z \tag{1.6}$$
$$q = c_1\, [m + p\,(r, z)], \quad m = q^\circ c_1^{-1}$$

Taking (1.5) and (1.6) into account we write the incompressibility equation in the form

$$\left[\left(\beta + \frac{\partial u}{\partial r}\right)\left(\beta^{-2} + \frac{\partial w}{\partial z}\right) - \frac{\partial w}{\partial r}\frac{\partial u}{\partial z}\right]\left(\beta + \frac{u}{r}\right) = 1 \tag{1.7}$$

Using the relationship

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$$\nabla \cdot [J (\nabla \mathbf{R}^T)^{-1}] = 0, \quad J = \det (\nabla \mathbf{R})$$

the equilibrium Eq.(1.1) for strain of the form (1.6) can be converted to the form

$$\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2} + \left(\beta + \frac{u}{r}\right)\left[\frac{\partial p}{\partial r}\left(\beta^{-2} + \frac{\partial w}{\partial z}\right) - \frac{\partial w}{\partial r}\frac{\partial p}{\partial z}\right]\right\}\mathbf{e}_r + \left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} + \left(\beta + \frac{u}{r}\right)\left[\frac{\partial p}{\partial z}\left(\beta + \frac{\partial u}{\partial r}\right) - \frac{\partial u}{\partial z}\frac{\partial p}{\partial r}\right]\right\}\mathbf{e}_z = 0 \tag{1.8}$$

The boundary conditions for the functions u, w, p are written thus

$$\left[\frac{\partial u}{\partial z} - \left(\beta + \frac{u}{r}\right)(m + p)\frac{\partial w}{\partial r}\right]\mathbf{e}_r + \left[\beta^{-2} + \frac{\partial w}{\partial z} + \left(\beta + \frac{u}{r}\right)(m + p)\left(\beta + \frac{\partial u}{\partial r}\right)\right]\mathbf{e}_z = 0, \quad z = \pm h$$
$$\frac{\partial w}{\partial r} - \left(\beta + \frac{u}{r}\right)(m + p)\frac{\partial u}{\partial z} = 0, \quad u = 0, \quad r = a \tag{1.9}$$

Introducing the dimensional quantities

$$r' = ra^{-1}, \quad z' = za^{-1}, \quad u' = ua^{-1}, \quad w' = wa^{-1}, \quad h' = ha^{-1}$$

we arrive from (1.7)-(1.9) at the following non-linear boundary value problem in $u'(r', z')$, $w'(r', z')$, $p(r', z')$ (we henceforth omit the primes):

$$l(x, D)\,\mathbf{v}(x) = \mathbf{f}(x, \mathbf{v}) \tag{1.10}$$

$$b(x, D)\,\mathbf{v}(x) = \mathbf{F}(x, \mathbf{v}), \quad z = \pm h, \quad r = 1 \tag{1.11}$$

Here

$$x \equiv (r, z) \quad \mathbf{v} \equiv (u, w, p), \quad \mathbf{f} \equiv (f_1, f_2, f_3)$$
$$F \equiv (F_1, F_2), \quad l(x, D) \equiv (l_{ij}(x, D))_{i,j=1,2,3}$$
$$b(x, D) \equiv (b_{ij}(x, D))_{i=1,2;\, j=1,2,3}$$
$$l_{11} = \partial_1^2 + r^{-1}\partial_1 + \partial_2^2 - r^{-2}, \quad l_{12} = 0$$
$$l_{13} = \beta^{-1}\partial_1, \quad l_{21} = 0, \quad l_{22} = \partial_1^2 + r^{-1}\partial_1 + \partial_2^2$$
$$l_{23} = \beta^2\partial_2, \quad l_{31} = -\beta^{-1}(\partial_1 + r^{-1}), \quad l_{32} = -\beta^2\partial_2, \quad l_{33} = 0$$
$$f_1 = (\beta + r^{-1}u)\,J(w, p) - \beta^{-2}r^{-1}u\partial_1 p$$
$$f_2 = (\beta + r^{-1}u)\,J(p, u) - \beta r^{-1}u\partial_2 p$$
$$f_3 = (\beta + r^{-1}u)\,J(u, w) + \beta r^{-1}u(\partial_2 w + \beta^{-3}\partial_1 u)$$
$$b_{11} = \partial_2, \quad b_{12} = \beta^{-3}\partial_1, \quad b_{13} = 0, \quad b_{21} = -\beta^{-3}(\partial_1 + r^{-1})$$
$$b_{22} = \partial_2, \quad b_{23} = \beta^2, \quad z = \pm h$$
$$b_{11} = 1, \quad b_{12} = 0, \quad b_{13} = 0, \quad b_{21} = \beta^{-3}\partial_2$$
$$b_{22} = \partial_1, \quad b_{23} = 0, \quad r = 1$$
$$F_1 = (\beta p - \beta^{-4}r^{-1}u + r^{-1}pu)\,\partial_1 w, \quad F_2 = (-\beta p + \beta^{-4}r^{-1}u - r^{-1}pu)\,\partial_1 u - \beta r^{-1}pu, \quad z = \pm h$$
$$F_1 = 0, \quad F_2 = (\beta p - \beta^{-4}r^{-1}u + r^{-1}pu)\,\partial_2 u, \quad r = 1$$
$$\partial_1 = \partial/\partial r, \quad \partial_2 = \partial/\partial z, \quad J(u, v) = \partial_1 u\partial_2 v - \partial_2 u\partial_1 v$$

The differential expressions l and b on the left-hand sides of (1.10) and (1.11) are linear but the expressions $\mathbf{f}$ and $\mathbf{F}$ do not contain linear components. The quantity β is a parameter in the boundary value problem (1.10) and (1.11).

2. It can be shown that the system (1.10) is elliptical according to Douglis-Nirenberg, while the boundary conditions (1.11) are additional /3, 4/, with the exception of the case when β satisfies the equation

$$\beta^9 + \beta^6 + 3\beta^3 - 1 = 0$$

As will be shown below, this case corresponds to the critical value of the loading parameter β for an infinite cylinder. We set the linear operator $A\mathbf{v} \equiv (l\mathbf{v}, b\mathbf{v})$, which we define in the Banach space

$$E_1 \equiv W_2^2(G) \oplus W_2^2(G) \oplus W_2{}^1{}_1(G); \quad G: \ (0 \leqslant r < 1, \ -h < z < h)$$

in correspondence to the left-hand sides of (1.10) and (1.11).

Then the domain of values of this operator belongs to the Banach space

$$E_2 \equiv L_2(G) \oplus L_2(G) \oplus W_2^1(G) \oplus W_2^{1/2}(\Gamma) \oplus W_2^{1/2}(\Gamma);$$
$$\Gamma: (r = 1,\ z = \pm h)$$

Here $W_2^2(G)$, $W_2^1(G)$ are Sobolev spaces, $L_2(G)$ is Lebesgue space, $W_2^{1/2}(\Gamma)$ is Slobodetskii space, and the symbol $\oplus$ denotes the direct sum of spaces.

We assume that the desired functions u, w, p belong to the space $W_2^3(G)$. Then it can be shown that the right-hand sides of (1.10) and (1.11) belong to E_2. This enables us to write the boundary value problem in the operator form

$$A\mathbf{v} = \tau, \quad \tau \equiv (\mathbf{f}, \mathbf{F}) \tag{2.1}$$

We write the necessary and sufficient condition for solvability of (2.1) by using the results of /4, 5/. In the case under consideration, it takes the following form after reduction

$$\int_0^1 \int_{-h}^{h} (f_1\psi_1 + f_2\psi_2 + f_3\psi_3)\, r\, dr\, dz - \int_0^1 (F_1\psi_1 + F_2\psi_2)\, r\, dr \,\Big|_{z=-h}^{h} + \int_{-h}^{h} [F_1(\partial_1\psi_1 - \beta^{-3}\partial_2\psi_2 + \beta^{-1}\psi_3) - F_2\psi_2]\, r\, dz \,\Big|_{r=0}^{1} = 0 \tag{2.2}$$

where $\psi = (\psi_1, \psi_2, \psi_3)$ are eigenvector-functions of the operator A that form a basis of the subspace of zeros of this operator.

3. To find the critical loading parameters for which bifurcation of the cylinder equilibrium occurs, we examine the linearized problem

$$A\mathbf{v} = 0 \tag{3.1}$$

Eq.(3.1) agrees with the neutral equilibrium equations for a cylinder as obtained in /2/. We will seek the eigenfunctions of problem (3.1) in the form

$$u = \sum_{n=1}^{\infty} u_n(z) J_1(k_n r), \quad w = w_0(z) + \sum_{n=1}^{\infty} w_n(z) J_0(k_n r), \quad p = p_0(z) + \sum_{n=1}^{\infty} p_n(z) J_0(k_n r) \tag{3.2}$$

where the k_n are determined from the condition

$$J_1(k_n) = 0 \tag{3.3}$$

and $J_0(k_n r)$, $J_1(k_n r)$ are Bessel functions. We substitute (3.2) into the left-hand side of (1.10) and solve them for u_n, w_n, p_n for $\mathbf{f} = 0$. We consequently obtain

$$\begin{aligned}
&u_n(z) = \beta_1 C_1 \operatorname{sh} \beta_2 z - \beta^3 C_2 \operatorname{sh} k_n z + \beta_1 C_3 \operatorname{ch} \beta_2 z - \beta^3 C_4 \operatorname{ch} k_n z, \quad w_n(z) = -\beta_1 C_1 \operatorname{ch} \beta_2 z + C_2 \operatorname{ch} k_n z - \\
&\beta_1 C_3 \operatorname{sh} \beta_2 z + C_4 \operatorname{sh} k_n z, \quad p_n(z) = C_1 \operatorname{sh} \beta_2 z + C_3 \operatorname{ch} \beta_2 z \\
&w_0(z) = C_5 + \beta^3 C_6 z^2/2, \quad p_0(z) = C_7 + C_6 z \\
&\beta_1 = \beta^5 (1 - \beta^6)^{-1} k_n^{-1}, \quad \beta_2 = \beta^{-3} k_n
\end{aligned} \tag{3.4}$$

The C_i $(i = 1, 2, \ldots, 7)$ in (3.4) are constants of integration.

Substituting expressions (3.2) into the boundary conditions (1.11) with zero right-hand sides, and taking (3.4) into account, we obtain a system of equations to determine C_i from which we find after reduction that $C_6 = C_7 = 0$ and we form two systems to determine $C_1, \ldots, C_4$

$$\begin{aligned}
&(1 + \beta^6) C_1 \operatorname{sh} \beta_2 h - 2C_2 \operatorname{sh} k_n h = 0 \\
&2C_1 \operatorname{ch} \beta_2 h - \beta^{-3}(1 + \beta^6) C_2 \operatorname{ch} k_n h = 0, \quad n = 1, 2, \ldots
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&(1 + \beta^6) C_3 \operatorname{ch} \beta_2 h - 2C_4 \operatorname{ch} k_n h = 0 \\
&2C_3 \operatorname{sh} \beta_2 h - \beta^{-3}(1 + \beta^6) C_4 \operatorname{sh} k_n h = 0, \quad n = 1, 2, \ldots
\end{aligned} \tag{3.6}$$

The constant C_5 remains undetermined. This is due to the fact that the boundary conditions allow cylinder displacement as an absolutely solid body in the z-axis direction.

Equating the determinants of systems (3.5) and (3.6) to zero for each n, we arrive at transcendental equations to determine the eigenvalues $\beta = \beta_0$ of problem (3.1), which are functions of k_n and h

$$\beta^{-3}(1 + \beta^6)^2 \operatorname{th} \beta_2 h = 4 \operatorname{th} k_n h \tag{3.7}$$

$$\beta^{-3}(1 + \beta^6)^2 \operatorname{cth} \beta_2 h = 4 \operatorname{cth} k_n h \tag{3.8}$$

If the cylinder is considered as a thick slab with middle surface $z = 0$, then (3.7) obtained earlier /2/ corresponds to the bending modes of slab equilibrium bifurcation when w is an even function and u, p are odd functions of the z coordinate. Eq.(3.8) corresponds to cylinder equilibrium modes symmetric with respect to the $z=0$ plane. Since $\beta \leqslant 1$, then $\tanh k_nh / \tanh \beta_2 h < 1$, and therefore, all roots of (3.7) are greater than the roots of (3.8) with the exception of the common root $\beta = 1$. Therefore, the critical value of the parameter β is found from (3.7).

Note that for an arbitrarily thick slab $(k_nh \to \infty)$ Eq.(3.7) reduces to the form

$$(\beta^3 - 1)(\beta^9 + \beta^6 + 3\beta^3 - 1) = 0 \tag{3.9}$$

Solving (3.9), we obtain $\beta_\infty = 0.6661$, from which it follows that an arbitrarily thick slab buckles for $\beta = \beta_\infty$. The critical value of the parameter $\beta = \beta_*$ is the maximum eigenvalue from the set of eigenvalues $\beta_0(k_n, h)$, where the quantity k_n is defined in (3.3). It can be shown that the eigenvalue β_0 is simple and takes the maximum value for $k_n = k_1 = 3.832$.

Below we present the critical values β_* obtained from (3.7) as well as the critical values β_*' found according to the classical theory of plate buckling /6/ for certain values of h

h	0.1	0.2	0.3	0.4	0.5
$\beta_*' \cdot 10^4$	9674	8695	7064	4780	1843
$\beta_* \cdot 10^4$	9669	8644	7484	6997	6809

It is seen that even for a fairly thick plate $(h \approx 0.3)$, the values of β_*' and β_* are close to one another.

4. To construct new equilibrium modes we will use the theory developed in /7/. Let β_0 be the eigenvalue of the operator A and λ the small parameter $(|\lambda| < \varepsilon)$. Then by setting $\beta = \beta_0 + \lambda$ it is possible to write (2.1) in the following form (A_0 is the operator A in which the quantity β is replaced by the eigenvalue β_0)

$$A_0\mathbf{v} = \tau - A\mathbf{v} + A_0\mathbf{v} \equiv \eta(\mathbf{v}), \quad \eta(\mathbf{v}) \equiv (\mathbf{t}, \mathbf{T}) \tag{4.1}$$

$$\mathbf{t} \equiv (f^1, f^2, f^3), \quad \mathbf{T} \equiv (F^1, F^2) \tag{4.2}$$

$$f^1 = f_1 - [\beta^{-1}]\, \partial_1 p, \quad f^2 = f_2 - [\beta^2]\, \partial_2 p$$

$$f^3 = \beta\beta_0^{-1} f_3 + [\beta^3]\beta_0^{-1}\partial_2 w$$

$$F^1 = F_1 - [\beta^{-3}]\, \partial_1 w, \quad F^2 = F_2 - [\beta^2]\, p + [\beta^{-3}]\, (\partial_1 u + r^{-1}u),$$

$$z = \pm h$$

$$F^1 = F_1 = 0, \quad F^2 = F_2 - [\beta^3]\, \partial_2 u, \quad r = 1$$

$$([\beta^k] = \beta^k - \beta_0^k)$$

Exactly as in /7/, we use the following notation: E_1^s is the subspace of zeros of the operator A_0 of dimensionality s, $E_1^{\infty-s}$ is the complement of the subspace E_1^s to E_1 and A_0^* is the contraction of the operator A_0 in $E_1^{\infty-s}$. Unlike A_0, the operator A_0^* will have a bounded inverse operator $\Gamma_0 = (A_0^*)^{-1}$ which we use in the construction of small solutions of (4.1).

Since the eigenvalues of the problem under consideration are always simple $(s = 1)$, we shall seek small solutions of (4.1) in the form of series

$$\mathbf{v} = \xi\boldsymbol{\psi} + \sum_{i=2}^{\infty} \mathbf{v}_{i0}\xi^i + \sum_{i=0}^{\infty} \xi^i \sum_{j=1}^{\infty} \mathbf{v}_{ij}\lambda^j \tag{4.3}$$

$$\mathbf{v}_{ij} \equiv (u_{ij}(r, z), w_{ij}(r, z), p_{ij}(r, z))$$

Here ξ is a formal parameter $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)$ is an eigenvector function of the operator A_0 corresponding to the eigenvalue β_0. Expanding terms containing β on the right-hand side of (4.1) in power series in λ, and the expressions containing u, w, p according to (4.3), we obtain

$$\eta(\mathbf{v}) = \sum_{i+j\geqslant 1} \eta_{ij}\xi^i\lambda^j, \quad \eta_{ij} \equiv (\mathbf{t}_{ij}, \mathbf{T}_{ij}) \tag{4.4}$$

$$\mathbf{t}_{ij} \equiv (f_{ij}^1, f_{ij}^2, f_{ij}^3), \quad \mathbf{T}_{ij} \equiv (F_{ij}^1, F_{ij}^2)$$

where f_{ij}^k, F_{ij}^l are coefficients of the expansion of the functions f^k, F^l $(k = 1, 2, 3;\ l = 1, 2)$, defined by the relationships (4.2).

Substituting (4.3) into (4.1) and equating the coefficients of identical powers $\xi^i\lambda^j$, and taking (4.4) into account, we obtain a recursion system to find $\mathbf{v}_{ij}$

$$A_0^*\mathbf{v}_{ij} = \eta_{ij} \tag{4.5}$$

from which we find

$$\mathbf{v}_{01} = 0, \quad \mathbf{v}_{11} = \Gamma_0\eta_{11}, \quad \mathbf{v}_{20} = \Gamma_0\eta_{20}, \; \ldots$$

To obtain the bifurcation equation from which the quantity ξ is determined, we substitute (4.3) into the solvability condition (2.2) for (4.1). We consequently obtain

$$\sum_{i=2}^{\infty} L_{i0}\xi^i + \sum_{i=0}^{\infty} \xi^i \sum_{j=1}^{\infty} L_{ij}\lambda^j = 0 \tag{4.6}$$

$$L_{ij} = \int_0^1 \int_{-h}^{h} (f_{ij}{}^1\psi_1 + f_{ij}{}^2\psi_2 + f_{ij}{}^3\psi_3)\, r\, dr\, dz -$$

$$\int_0^1 (F_{ij}{}^1\psi_1 + F_{ij}{}^2\psi_2)\, r\, dr\, \Big|_{z=-h}^{h} - \int_{-h}^{h} F_{ij}{}^2\psi_2\, dz\, \Big|_{r=1}$$

It is taken into account in (4.6) that $F^1 = 0$ for $r = 1$.

5. We construct the inverse operator Γ_0. To do this we **consider** the equation

$$A_0{}^*\mathbf{v} = (G_1, G_2, G_3, g_1, g_2) \tag{5.1}$$

Let us right-hand side of (5.1) be representable in the form of series in the orthogonal functions $J_0(k_m r)$ and $J_1(k_m r)$ (here and henceforth the summation is over m between 1 and ∞)

$$G_1 = \Sigma G_{1m}(z)\, J_1(k_m r), \quad G_2 = G_{20}(z) + \Sigma G_{2m}(z)\, J_0(k_m r), \tag{5.2}$$

$$G_3 = G_{30}(z) + \Sigma G_{3m}(z)\, J_0(k_m r)$$

$$g_1 = \Sigma g_{1m}(z)\, J_1(k_m r), \quad g_2 = g_{20}(z) + \Sigma g_{2m}(z)\, J_0(k_m r),$$

$$z = \pm h$$

$$g_1 = g_2 = 0, \quad r = 1$$

where $G_{20}(z)$, $g_{20}(z)$ are even functions and k_m satisfies the condition $J_1(k_m) = 0$.
We will seek the solution of (5.1) in the form

$$u = \Sigma u_m(z)\, J_1(k_m r), \quad w = w_0(z) + \tag{5.3}$$

$$\Sigma w_m(z)\, J_0(k_m r), \quad p = p_0(z) + \Sigma p_m(z)\, J_0(k_m r)$$

Substituting (5.2) into (5.1) and equating the left- and right-hand sides for identical $J_0(k_m r)$ and $J_1(k_m r)$, we obtain a system of ordinary differential equations to determine u_m, w_m, p_m, w_0, p_0 with boundary conditions for $z = \pm h$. The solution of this system will be

$$u_m(z) = C_{1m}\beta_3 \operatorname{sh}\beta_4 z + C_{2m}\beta_3 \operatorname{ch}\beta_4 z - \tag{5.4}$$

$$C_{3m}\beta_0{}^3 \operatorname{sh} k_m z - C_{4m}\beta_0{}^3 \operatorname{ch} k_m z + \int_0^z I_1{}^2(\tau, \beta_4) \operatorname{ch} k_m(z-\tau)\, d\tau +$$

$$\beta_0^{-3} \int_0^z I_2{}^2(\tau, \beta_4) \operatorname{sh} k_m(z-\tau)\, d\tau + \beta_0^{-2} I_3{}'(z, \beta_4)$$

$$w_m(z) = -C_{1m}\beta_3 \operatorname{ch}\beta_4 z - C_{2m}\beta_3 \operatorname{sh}\beta_4 z + C_{3m} \operatorname{ch} k_m z +$$

$$C_{4m} \operatorname{sh} k_m z - \int_0^z I_1{}^1(\tau, \beta_4) \operatorname{ch} k_m(z-\tau)\, d\tau -$$

$$\beta_0^{-3} \int_0^z I_2{}^1(\tau, \beta_4) \operatorname{sh} k_m(z-\tau)\, d\tau - \beta_0^{-2} I_3{}^2(z, \beta_4)$$

$$p_m(z) = C_{1m} \operatorname{sh}\beta_4 z + C_{2m} \operatorname{ch}\beta_4 z + \beta_0^{-2} I_1{}^1(z, \beta_4) +$$

$$\beta_0^{-2} I_2{}^2(z, \beta_4) + k_m\beta_5\beta_0^{-7} I_3{}^1(z, \beta_4) + \beta_0^{-4} G_{3m}(z)$$

$$w_0(z) = C_{10} - \beta_0^{-2} \int_0^z G_{30}(\tau)\, d\tau$$

$$p_0(z) = \beta_0^{-2} \Big[g_{20}(h) - \int_0^h G_{20}(\tau)\, d\tau + \int_0^z G_{20}(\tau)\, d\tau + \beta_0^{-2} G_{30}(z)\Big]$$

$$C_{1m} = \beta_7\Delta_1^{-1} (2Q_m{}^+ \operatorname{sh} k_m h + \beta_8 R_m{}^- \operatorname{ch} k_m h)$$

$$C_{3m} = k_m^{-1}\Delta_1^{-1} (\beta_6 Q_m{}^+ \operatorname{sh}\beta_4 h + 2R_m{}^- \operatorname{ch}\beta_4 h)/2, \quad m \neq n$$

$$C_{1n} = 0, \quad C_{3n} = R_n{}^-/(4k_n \operatorname{sh} k_n h)$$

$$C_{2m} = \beta_7\Delta_2^{-1} (2Q_m{}^- \operatorname{ch} k_m h + \beta_8 R_m{}^+ \operatorname{sh} k_m h)$$

$$C_{4m} = k_m^{-1}\Delta_2^{-1} (\beta_6 Q_m{}^- \operatorname{ch}\beta_4 h + 2R_m{}^+ \operatorname{sh}\beta_4 h)/2$$

$$Q_m^{\pm} = Q_m(h) \pm Q_m(-h), \quad R_m^{\pm} = R_m(h) \pm R_m(-h)$$
$$Q_m(z) = g_{1m}(z) - 2\beta_5^{-1}[I_1^2(h, \beta_4) + I_2^1(h, \beta_4)] -$$
$$2k_m\beta_0^{-5}I_3^2(h, \beta_4) + \beta_6\beta_5^{-1}[I_1^2(h, k_m) + \beta_0^{-3}I_2^1(h, k_m)]$$
$$R_m(z) = g_{2m}(z) + \beta_6\beta_5^{-1}[I_1^1(h, \beta_4) + I_2^2(h, \beta_4)] +$$
$$k_m\beta_6\beta_0^{-5}I_3^1(h, \beta_4) - 2\beta_5^{-1}[\beta_0^3 I_1^1(h, k_m) + I_2^2(h, k_m)]$$
$$I_k^1(l, \alpha) = \int_0^l \operatorname{sh}\alpha(l-\sigma)G_{km}(\sigma)\,d\sigma$$
$$I_k^2(l, \alpha) = \int_0^l \operatorname{ch}\alpha(l-\sigma)G_{km}(\sigma)\,d\sigma$$
$$\Delta_1 = \Delta(\beta_4, k_m), \quad \Delta_2 = \Delta(k_m, \beta_4)$$
$$\Delta(\alpha, l) = 4\operatorname{ch}\alpha h \operatorname{sh} lh - \beta_6\beta_8 \operatorname{sh}\alpha h \operatorname{ch} lh$$
$$\beta_3 = k_m^{-1}\beta_0^5(1-\beta_0^6)^{-1}, \quad \beta_4 = k_m\beta_0^{-3}, \quad \gamma_5 = 1 - \beta_0^6$$
$$\beta_6 = 1 + \beta_0^6, \quad \beta_7 = \beta_5\beta_0^{-2}/2, \quad \beta_8 = \beta_6\beta_0^{-3}$$

C_{10} is an arbitrary constant, and n is the number of the root of (3.3) corresponding to the value β_0.

6. To find the coefficients L_{ij} of the bifurcation Eq.(4.6), we write down the first coefficients of the expansion of the right-hand side of (4.1)

$$f_{01}^1 = f_{01}^2 = f_{01}^3 = F_{01}^1 = F_{01}^2 = 0 \tag{6.1}$$

$$f_{11}^1 = \beta_0^{-2}\partial_1\psi_3, \quad f_{11}^2 = -2\beta_0\partial_2\psi_3, \quad f_{11}^3 = 3\beta_0\partial_2\psi_2 \tag{6.2}$$
$$F_{11}^1 = 3\beta_0^{-4}\partial_1\psi_2, \quad F_{11}^2 = 3\beta_0^{-1}\partial_2\psi_2 - 2\beta_0\psi_3, \quad z = \pm h$$
$$F_{11}^1 = F_{11}^2 = 0, \quad r = 1$$

$$f_{20}^1 = \beta_0\partial_2\psi_3\partial_1\psi_2 + \beta_0^{-2}\partial_1\psi_3\partial_1\psi_1 \tag{6.3}$$
$$f_{20}^2 = \beta_0\partial_1\psi_3\partial_2\psi_1 + \beta_0^4\partial_2\psi_3\partial_2\psi_2$$
$$f_{20}^3 = \beta_0^{-2}r^{-1}\psi_1\partial_1\psi_1 - \beta_0^4\partial_2\psi_2\partial_2\psi_2 - \beta_0\partial_1\psi_2\partial_2\psi_1$$
$$F_{20}^1 = \beta_0\psi_3\partial_1\psi_2 + \beta_0^{-1}r^{-1}\psi_1\partial_2\psi_1, \quad F_{20}^2 = \beta_0^4\psi_3\partial_2\psi_2 +$$
$$\beta_0^{-4}r^{-1}\psi_1\partial_1\psi_1, \quad z = \pm h$$
$$F_{20}^1 = F_{20}^2 = 0, \quad r = 1$$
$$\psi_1 = C(A_n \operatorname{sh}\beta_9 z - \beta_0^3 B_n \operatorname{sh} k_n z)J_1(k_n r)$$
$$\psi_2 = C(-A_n \operatorname{ch}\beta_9 z + B_n \operatorname{ch} k_n z)J_0(k_n r), \quad \psi_3 =$$
$$C \operatorname{sh}\beta_9 z J_0(k_n r)$$
$$\beta_9 = \beta_0^{-3}k_n, \quad A_n = \beta_0^5/(k_n\beta_5),$$
$$B_n = \beta_0\beta_6 \operatorname{sh}\beta_9 h/(2\beta_5 k_n \operatorname{sh} k_n h)$$

where C is an arbitrary fixed constant.

It follows from (6.1) that $L_{01}=0$. Substituting (6.2) into (4.7) for $i = 1, j = 1$ we obtain the following expression for L_{11} after simple reduction:

$$L_{11} = {}^3/_2\beta_0^3\beta_5^{-1}[\beta_0^3\beta_6^{-1}k_n^{-1}(1 - 3\beta_0^6)\operatorname{sh} 2\beta_9 h + 2h]J_0^2(k_n) \tag{6.4}$$

It is seen from (6.3) that the functions $\psi_2, f_{20}^1, f_{20}^3, F_{20}^2$ are even in z while the functions $\psi_1, \psi_3, f_{20}^2, F_{20}^1$ are odd in z. It hence follows that $L_{20} = 0$. Writing down expressions for f_{30}^k, F_{30}^l $(k = 1, 2, 3;\ l = 1, 2)$ and substituting them into (4.6) for $i = 3, j = 0$, we obtain after the reduction of similar terms

$$L_{30} = 2\int_{-h}^{h}\int_0^1 \{(\psi_1\psi_3 + \beta_0 r p_{20})[J(\psi_1, \psi_2) + \beta_0^{-3}r^{-2}\psi_1^2] + \tag{6.5}$$
$$\beta_0 r\psi_3[J(\psi_1, w_{20}) + J(u_{20}, \psi_2) - 2\beta_0^{-3}r^{-2}\psi_1 u_{20}]\}\,dr\,dz -$$
$$2\beta_0^{-4}\int_0^1 \psi_1(w_{20}\partial_1\psi_1 - u_{20}\partial_1\psi_2)\,dr\,\Big|_{z=-h}^{h}$$

To determine $v_{20} \equiv (u_{20}, w_{20}, p_{20})$ we use the inverse operator Γ_0. We use the following notation: $G_k = f_{20}^k$, $g_l = F_{20}^l$ $(k = 1, 2, 3;\ l = 1, 2)$; $u = u_{20}$, $w = w_{20}$, and $p = p_{20}$; then $\mathbf{v}_{20}$ will be determined from relationships (5.3) and (5.4) in which, by virtue of (6.3),

$$G_{20} = G_{30} = g_{20} = 0, \quad p_0 = 0, \quad w_0 = \text{const}$$
$$C_{1m} = C_{3m} = 0 \quad \text{for any} \quad m$$

Since the solution $w = \text{const}$ corresponds to displacement of the cylinder as a solid body it is possible to set $w_0 = 0$. For the same reason there is no analogous component in ψ_2. Therefore, the bifurcation Eq. (4.6) approximately takes the form

$$L_{30}\xi^3 + L_{11}\xi\lambda \approx 0$$

It hence follows that $\xi = \pm (\lambda\Lambda)^{1/2} + o(\lambda^{1/2})$ $(\Lambda = -L_{11}L_{30}^{-1})$, and the solution (4.3) of (4.1) is written in the form

$$\mathbf{v} = \pm (\lambda\Lambda)^{1/2}\psi + \lambda\Lambda\mathbf{v}_{20} \pm (\lambda\Lambda)^{1/2}\lambda\mathbf{v}_{11} + o(\lambda)$$

Two new solutions will occur depending on the sign of the expression Λ in one of the semicircles $(\beta_0 - \varepsilon, \beta_0)$ or $(\beta_0, \beta_0 + \varepsilon)$ while there will be no other new solutions.

A numerical investigation of the coefficients L_{11}, L_{30} in (6.4) and (6.5) showed that L_{11} is always greater than zero while L_{30} can change sign depending on the value of $k_n h$. Certain values of the coefficients of the bifurcation equation are presented below for $C = 1$

h	0.1	0.2	0.3	0.5
L_{11}	$2.45\cdot10^{-2}$	$6.49\cdot10^{-2}$	0.730	$4.18\cdot10^{2}$
L_{30}	$1.17\cdot10^{-5}$	$-8.89\cdot10^{-4}$	-0.202	$-2.28\cdot10^{6}$

In the case of the maximum eigenvalue $L_{30} > 0$ for $h < 0.1135$ and $L_{30} < 0$ for $h > 0.114$. It hence follows that for comparatively thin plates $(h < 0.1135)$ new solutions, different from the trivial one, occur only for loads exceeding the minimal critical load, which is in agreement with the results of the two-dimensional theory of thin plates /8/. For thick slabs, new solutions close to the fundamental occur only for loads less than the critical (a qualitative difference from the behaviour of thin plates).

An analgous phenomenon of a qualitative change in the nature of the bifurcation as the thickness increases was found in /5/ in the problem of the post-critical behaviour of a thick-walled pipe of a compressible semilinear material. This provides a basis for concluding that the fact of a qualitative distinction between the post-critical behaviour of thick-walled and thin-walled structures is independent of the properties of the material. Meanwhile it is clear that the value of the relative thickness at which a qualitative change in the bifurcation pattern occurs will be different for different materials.

The noted features in the behaviour of thick-walled elastic bodies can be expected in other problems also, for instance, in the still uninvestigated plane problem of the post-critical behaviour of a compressed rectangular bar.

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